Formal Analysis of Fractional Order Systems in HOL

Umair Siddique Osman Hasan

System Analysis and Verification (SAVE) Lab
National University of Sciences and Technology (NUST)
Islamabad, Pakistan

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Outline

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Fractional Order Systems

- Physical systems are usually modeled with integral and differential equations

$$D^n f(x) = \frac{d^n}{dx^n} f(x) = \frac{d}{dx}(\frac{d}{dx} \cdots \frac{d}{dx}(f(x)) \cdots))$$

$$\int \int \cdots \int f(x_1, x_2, \cdots x_n) dx_1, dx_2 \cdots dx_n$$

- Are these traditional concepts sufficient?

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  - Cannot be modeled using an integer order Differential Equation

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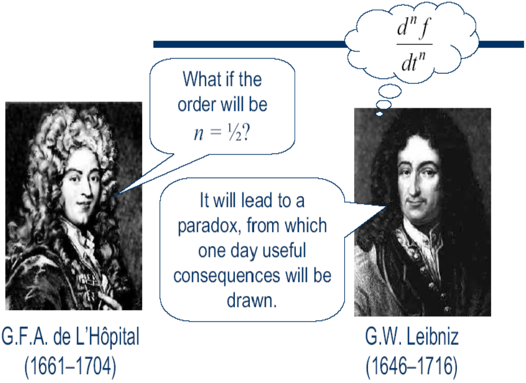
- Example
  - Resistoductance: Exhibits intermediate behavior between a Resistor ($v = iR$) and an Inductor ($v = L\frac{di}{dt}$)
  - Cannot be modeled using an integer order Differential Equation
- Fractional Order Systems involve derivatives and integrals of non integer order (Fractional Calculus)

Fractional order Calculus

- Fractional Calculus was born in 1695



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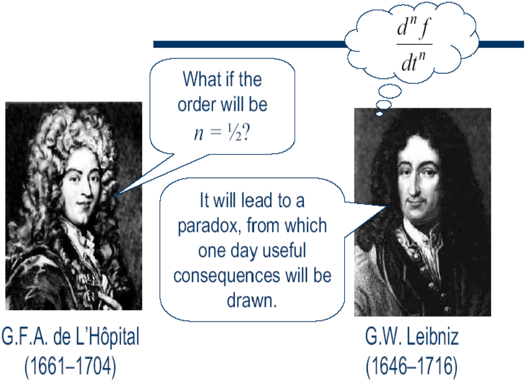

- Why a paradox?
- Useful Consequences?

Fractional order Calculus - Why a Paradox?

- Analogous to fractional exponents

$$x^3 = x \bullet x \bullet x$$

$$x^{3.7} = ?$$

$$x^{\pi} = ?$$

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- Integrals and Derivatives are certainly more complex than multiplication
- Fractional Integrals and Derivatives can be defined in numerous ways
- Fractional Calculus started off as a study for the best minds in mathematics
  - Leibniz, Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Riemann

Mathematical Definitions of Fractional Calculus

Mathematical Definitions of Fractional Calculus

Definition (Euler's Fractional Derivative for Power Function x^p)

$$D^0 x^p = x^p, D^1 x^p = p x^{p-1}, D^2 x^p = p(p-1)x^{p-2} \cdots$$

can be generalized as follows:

$$D^n x^p = \frac{p!}{(p-n)!} x^{p-n}; \qquad n : integer$$

Gamma function generalizes the factorial for all real numbers

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

Thus

$$D^n x^p = \frac{\Gamma(p+1)}{\Gamma(p-n+1)} x^{p-n}; \qquad n : real$$

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- Limited Scope (Only caters for power functions $f(x) = x^y$)

Mathematical Definitions of Fractional Calculus

Definition (Riemann-Liouville (RL) Fractional Integration)

$$J_a^v f(x) = \int\int \cdots \int_a^t f(x)dx = \frac{1}{\Gamma(v)} \int_a^x (x-t)^{v-1} f(t)dt$$

Definition (Riemann-Liouville Fractional Differentiation)

$$D^v f(x) = (\frac{d}{dx})^{\lceil v \rceil} J_a^{\lceil v \rceil - v} f(x)$$

where v is the order and $\lceil v \rceil$ is its *ceiling* (largest and closest integer).

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where v is the order and $\lceil v \rceil$ is its *ceiling* (largest and closest integer).

- General definition that caters for all functions that can be expressed in a closed mathematical form
- Usage requires expertise and rigorous mathematical analysis

Mathematical Definitions of Fractional Calculus

Definition (Grünwald-Letnikov (GL) Fractional Diffintegral)

$$ {}_cD_x^v f(x) = \lim_{h \to 0} h^{-v} \sum_{k=0}^{[\frac{x-c}{h}]} (-1)^k \binom{v}{k} f(x - kh) $$

where $\binom{v}{k}$ represents the binomial coefficient expressed in terms of the Gamma function

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- $(0 < v)$: Fractional Differentiation
- $(v < 0)$: Fractional Integration

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- $(0 < v)$: Fractional Differentiation
- $(v < 0)$: Fractional Integration
- Facilitates Numerical Methods based computerized analysis
    - Approximate solutions due to the infinite summation involved

Fractional order Calculus

- Paradox Resolved!
  - Most of the Mathematical Fractional Calculus theory was developed prior to the turn of the 20th century
- Useful Consequences?
  - First book on modeling Engineering systems using Fractional Calculus was published in 1974 by Oldham and Spanier
  - Recent monographs and symposia proceedings have highlighted the application of Fractional Calculus in
    - Continuum Mechanics
    - Signal Processing
    - Electro-magnetics
    - Control Engineering
    - Electronic Circuits
    - Biological Systems

Analysis of Fractional Order Systems

- Fractional order Systems are widely used in safety-critical domains like medicine and transportation
  - Example: Cardiac tissue electrode interface
- Analysis inaccuracies may even result in the loss of human lives

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- Usage of Fractional Calculus guarantees correct models
- What about the accuracy of Analysis techniques?

Analysis of Fractional Order Systems: Comparison

Criteria	Paper-and-Pencil Proof	Simulation	Automated Formal Methods (MC, ATPs)	Higher-order-logic Theorem Proving
Expressiveness				
Scalability				
Accuracy				
FOS Fundamentals				
Automation				

Analysis of Fractional Order Systems: Comparison

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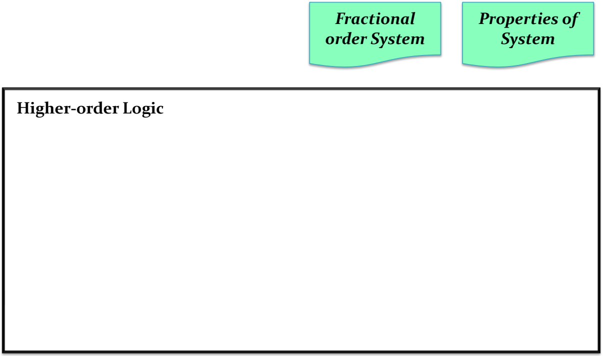
Outline

Proposed Framework

Proposed Framework



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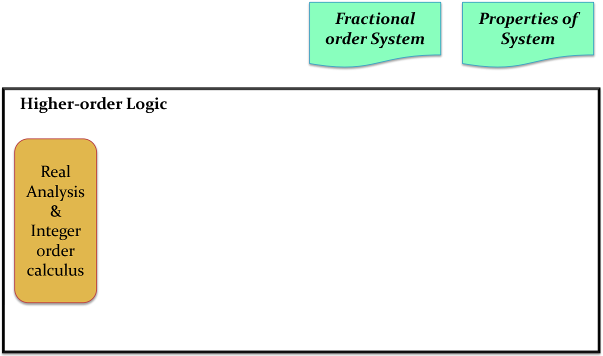

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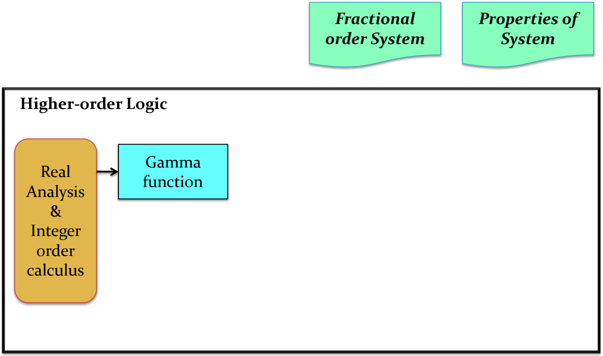

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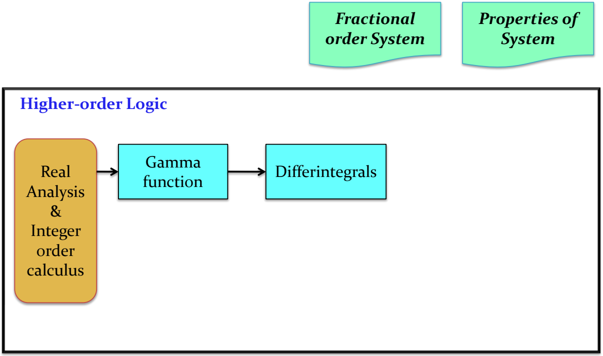

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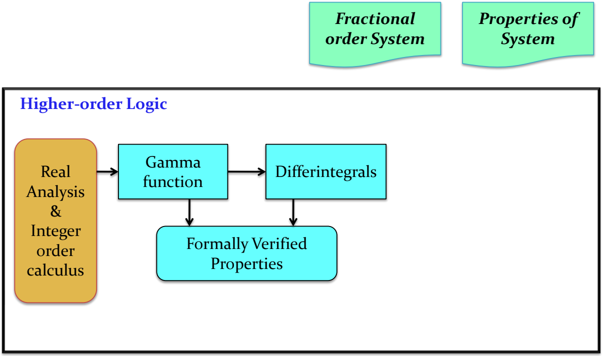

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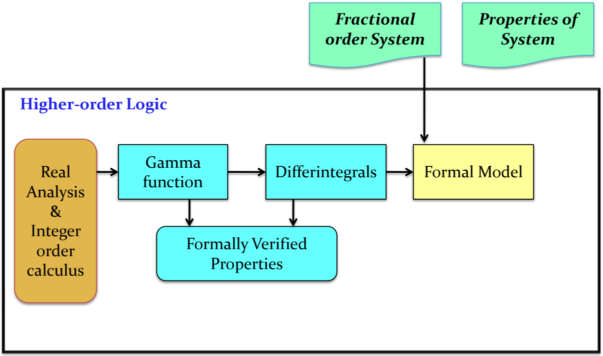

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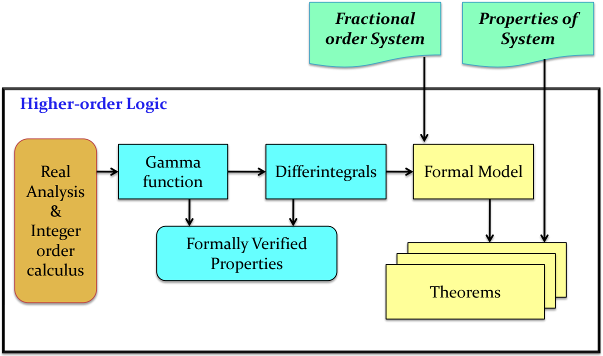

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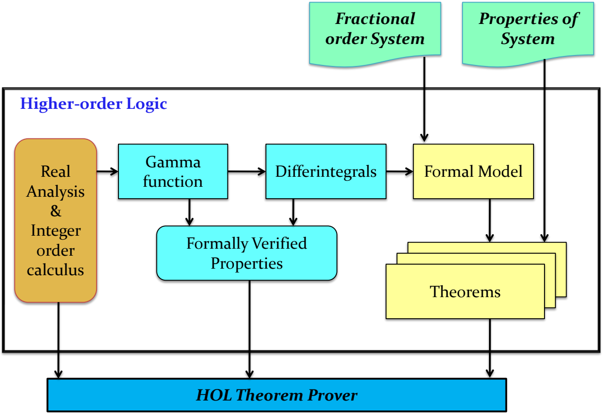

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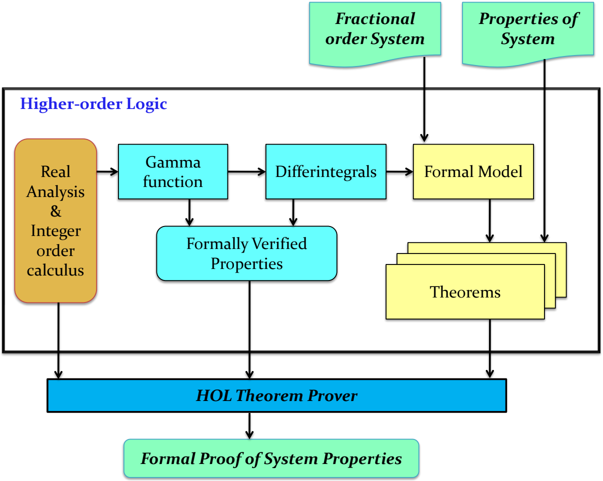

HOL4 Theorem Prover

- Higher-order-logic Theorem Prover developed at the University of Cambridge
- Its core consists of
  - 5 fundamental axioms (facts)
  - 8 Inference rules
- Soundness is assured as every new theorem must be created from
  - The basic axioms and primitive inference rules
  - Any other already proved theorems (Theory Files)

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  - The basic axioms and primitive inference rules
  - Any other already proved theorems (Theory Files)
- The availability of Harisson's seminal work on Real analysis and Integer order Calculus has been the primary motivation for this choice

Outline

Formalization of Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

- The integrand $t^{z-1}e^{-t}$ becomes unbounded on the lower limit $(t = 0)$ for $z < 1$
- The upper limit ∞ is undefined

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Definition

⊢ ∀ z. gamma z =
  lim(λn.(lim(λb. $\int_{\frac{1}{2^n}}^{b}$ t rpow (z-1)exp(-t) dt))

Formal Verification of Gamma function properties

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$$\Gamma(z+1) = z\Gamma(z)$$

Theorem: Pseudo-recurrence Relation

$\vdash \forall$ `z . (0 < z)` $\Longrightarrow$ `(gamma (z + 1)= z gamma (z))`

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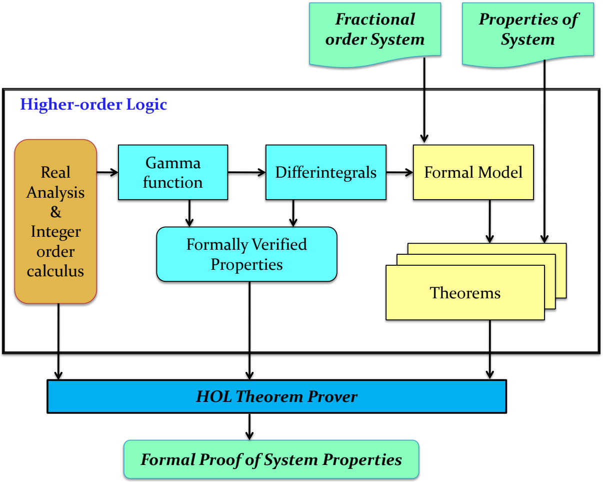
- The paper-and-pencil based proof is based on the integration-by-parts property
- We also had to utilize the concepts of limits of a real sequence, differentiability and integrability
- The formal proof required 10 main lemmas. e.g.,
  - $(\forall n.\exists k.(\lambda b. \int_{\frac{1}{2^n}}^{b} t^{z-1}e^{-t}dt) \longrightarrow k)$
  - $(\forall b.\exists p.(\lambda n. \int_{\frac{1}{2^n}}^{b} t^{z-1}e^{-t}dt) \longrightarrow p)$
- It took approximately 2000 lines of ML code

Formally Verified Properties of Gamma Function

Property	HOL Formalization
Pseudo-Recurrence Relation	$\vdash \forall$ `z.(0 < z)` $\Longrightarrow$ `(gamma (z + 1)= z gamma (z))`
Functional Equation	$\vdash$ `gamma 1 = 1`
Factorial Generalization	$\vdash \forall$ `n` $\in \mathbb{N}$. `gamma(n + 1) = n!`

Proposed Framework



Formalization of Fractional Integration

- We follow Riemann-Liouville Definition

$$J_a^v f(x) = \frac{1}{\Gamma(v)} \int_a^x (x-t)^{v-1} f(t) dt$$

- The integrand $(x-t)^{v-1} f(t)$ becomes undefined on upper limit (x) of integration

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$$J_a^v f(x) = lim_{n \to \infty} \left(\frac{1}{\Gamma(v)} \int_a^{x - \frac{1}{2^n}} (x-t)^{v-1} f(t) dt \right)$$

Definition

$\vdash \forall$ f v a x.frac_int f v a x = if (v = 0) then f else lim(λn.$\frac{1}{\text{gamma v}}$($\int_a^{x-\frac{1}{2^n}}$ ((x - t) rpow (v-1)) f(t) dt)

Formalization of Fractional Differentiation

$$D^v f(x) = (\frac{d}{dx})^{\lceil v \rceil} J_a^{\lceil v \rceil - v} f(x)$$

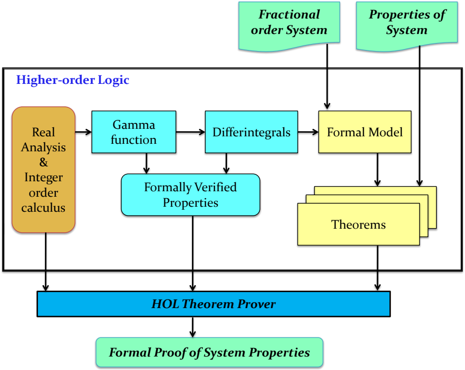
Definition

```
⊢ ∀ f v a x.  frac_diff f v a x =
   n_order_deriv (clg v) (frac_int f (clg v - v) a x)
```

Formally Verified Properties of Differintegrals

Property	**HOL Formalization**
Identity	⊢ ∀ f a x. (a < x) ⟹ (frac_int f 0 a x = f)∧ (frac_diff f 0 a x = f)
Generalized_Integral	⊢ ∀ f a x v ∈ N. (a < x)∧ (1 < v) ⟹ frac_int f v a x = lim(λn. $\frac{1}{\texttt{(v-1)!}} \int_a^{x-\frac{1}{2^n}}$ (x - t) rpow (v-1)f(t) dt)
frac_int Linearity	⊢ ∀ f v x a b. (frac_exists f x v)∧ (frac_exists g x v) ⟹ frac_int (a f + b g) v 0 x = a(frac_int f v 0 x)+ b(frac_int g v 0 x)
frac_diff Linearity	⊢ ∀ f v x a b. (frac_exists f x v)∧ (frac_exists g x v)∧ (∀ m. (m <= clg v) ⇒ (n_order_deriv m (frac_int f v 0 x)) differentiable x)∧ (∀ m. (m <= clg v) ⇒ (n_order_deriv m (frac_int g v 0 x)) differentiable x)⟹ (frac_diff (a f + b g) v 0 x = a(frac_diff f v 0 x)+ b(frac_diff g v 0 x))

HOL Formalization of Fractional Caclulus



- The formalization took around 7500 lines of ML code and approximately 600 man hours

Outline

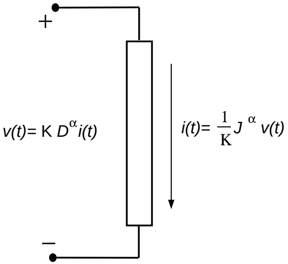
Case Studies

Case Studies

- We apply our framework to analyze three real world fractional order systems
  - Resistoductance
  - Fractional Differentiator circuit
  - Fractional Integrator circuit

Resistoductance

- An electrical component with characteristics between ohmic resistor and an Inductor



- $\alpha = 0$: Purely resistive behavior with K = R ohms
- $\alpha = 1$ Purely inductive behavior with K = L henrys

Formal Model of Resistoductance

- The governing current-voltage relationship is given as follows:

$$i(t) = \frac{1}{K} J^{\alpha} v(t)$$

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Definition (Resistoductance Current)

```
⊢ ∀ K v_i alpha x.
   i_t K v_i alpha x = (1/K)frac_int v_i(t) alpha 0 x
```

- v_i = Input voltage
- i_t = Resistoductance current
- alpha = Order of integration

Verification of Resistoductance properties

Verification of Resistoductance properties

i_t for constant voltage V_0

```
⊢ ∀ K V_0 alpha x.  (0 < x) ∧ (0 < alpha) ⟹
    (i_t K V_0 alpha x =
        (1/(K Gamma (alpha + 1))) (V_0(x rpow alpha)))
```

Verification of Resistoductance properties

i_t for constant voltage V_0

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   (i_t K V_0 alpha x =
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```

Theorem: Special Cases for i_t

```
⊢ ∀ x.  (0 < x) ⟹
      (alpha = 0) ⇒ i_t K V_0 alpha x = V_0 / K ∧
      (alpha = 1) ⇒ i_t K V_0 alpha x = (V_0 / K) x
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Verification of Resistoductance properties

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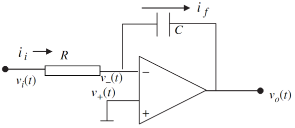
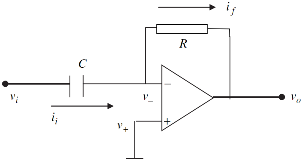
```

- Proof heavily relies upon the formally verified properties of Gamma function and Differintegrals
- 350 lines of HOL code

Fractional integrator and differentiator circuits



(a)



(b)

- Used in fractional-order PID and PI controllers
- Offer more flexibility for gain adjustment

Formal Models

Formal Models

- The output voltage equations for a fractional integrator

$$v_o(t) = -\frac{1}{RC} J^{\mu} v_i(t)$$

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Definition (Fractional Order Integrator)

```
⊢ ∀ R C v_i mu x.  v_I_0 R C v_i mu x =
           -(1/RC)frac_int v_i(t) mu 0 x
```

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- The output voltage equations for a fractional differentiator

$$v_0(t) = -RC D^{\mu} v_i(t)$$

Formal Models

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```

- The output voltage equations for a fractional differentiator

$$v_0(t) = -RC D^{\mu} v_i(t)$$

Definition (Fractional Order Differentiator)

```
⊢ ∀ R C v_i mu x.  v_D_0 R C v_i mu x=
      -(RC)frac_diff v_i(t) mu 0 x
```

Formal Analysis: For Unit Step signal

Formal Analysis: For Unit Step signal

Theorem: Output of Fractional Integrator Circuit

$\vdash \forall$ R C mu x. (0 < x) $\wedge$ (0 < mu) $\wedge$ (mu < 1) $\Longrightarrow$
(v_I_0 R C (unit t) mu x =
-1/(RC Gamma (mu + 1)) (x rpow mu)

Formal Analysis: For Unit Step signal

Theorem: Output of Fractional Integrator Circuit

```
⊢ ∀ R C mu x.  (0 < x) ∧ (0 < mu) ∧ (mu < 1) ⟹
  (v_I_0 R C (unit t) mu x =
      -1/(RC Gamma (mu + 1)) (x rpow mu)
```

Theorem: Output of Fractional Differentiator Circuit

```
⊢ ∀ R C mu x.  (0 < x) ∧ (0 < mu) ∧ (mu < 1) ⟹
  (v_D_0 R C (unit t) mu x =
      (-(RC (Gamma (1 - mu)))(x rpow -mu))
```

Formal Analysis: For Unit Step signal

Theorem: Output of Fractional Integrator Circuit

```
⊢ ∀ R C mu x.  (0 < x) ∧ (0 < mu) ∧ (mu < 1) ⟹
  (v_I_0 R C (unit t) mu x =
      -1/(RC Gamma (mu + 1)) (x rpow mu)
```

Theorem: Output of Fractional Differentiator Circuit

```
⊢ ∀ R C mu x.  (0 < x) ∧ (0 < mu) ∧ (mu < 1) ⟹
  (v_D_0 R C (unit t) mu x =
      (-(RC (Gamma (1 - mu)))(x rpow -mu))
```

- The proof relies heavily upon the proposed formalization
- 400 lines of HOL code
- Approximately 2.5 man-hours

Outline

Conclusions

- Summary
  - Formalization of Gamma function and Differintegrals
  - Formal Analysis of Fractional order Systems

Conclusions

- Summary
  - Formalization of Gamma function and Differintegrals
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- Future Work
  - Enriching the library of the formally verified Fractional Calculus properties
    - Law of Exponents
    - Relationship with the Beta function
  - Development of the current framework using Complex Numbers
  - More Case Studies
    - Fractional Electromagnetic Systems (Fractional Rectangular Waveguides)

Thank You!

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