IC3: Where Monolithic and Incremental Meet

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Outline

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1. Proving Invariants by Induction
   - Induction for Transition Systems
   - Strengthening
   - Relative Induction

2. IC3
   - Basic Algorithm
   - Examples
   - Efficiency

Finite-State Transition Systems

IC3 works on a symbolic representation of a system:

$$S : (\bar{i},\ \bar{x},\ I(\bar{x}),\ T(\bar{i},\bar{x},\bar{x}'))$$

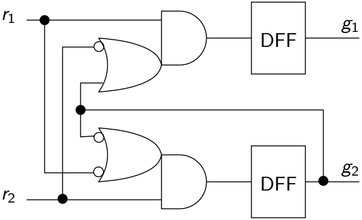
- $\bar{i}$: primary inputs
- $\bar{x}$: state variables
- $\bar{x}'$: next state variables
- $I(\bar{x})$: initial states
- $T(\bar{i},\bar{x},\bar{x}')$: transition relation

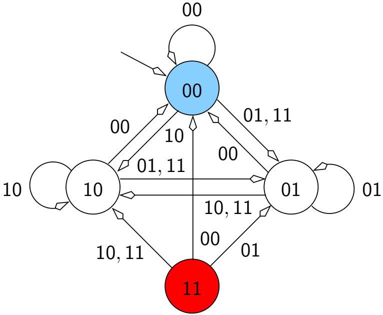
Invariance Properties

IC3 proves (or refutes) invariants

- Prove that every reachable state satisfies $P(\overline{x})$
  - P is a propositional formula
- Checking safety properties is reduced to checking invariance properties

Mutual Exclusion for a Simple Arbiter





$$I(\overline{g}) = \neg g_1 \wedge \neg g_2$$

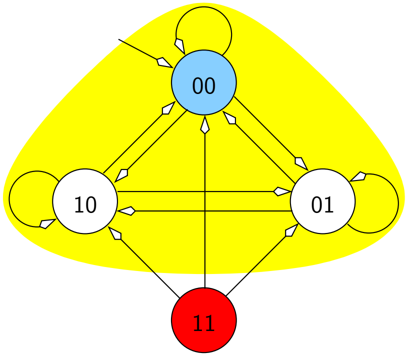
$$\exists r_1, r_2 \,.\, T(\overline{r}, \overline{g}, \overline{g}') = \neg g_1' \vee \neg g_2'$$

$$P(\overline{g}) = \neg g_1 \vee \neg g_2$$

Inductive Proofs for Transition Systems

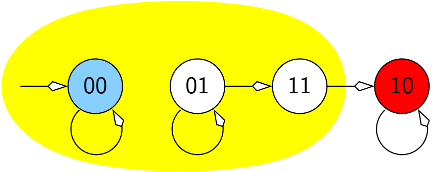
- Prove initiation (base case)
  - $I(\overline{x}) \Rightarrow P(\overline{x})$
  - All initial states satisfy P
  - $(\neg g_1 \wedge \neg g_2) \Rightarrow (\neg g_1 \vee \neg g_2)$
- Prove consecution (inductive step)
  - $P(\overline{x}) \wedge T(\overline{i}, \overline{x}, \overline{x}') \Rightarrow P(\overline{x}')$
  - All successors of states satisfying P satisfy P
  - $(\neg g_1 \vee \neg g_2) \wedge (\neg g_1' \vee \neg g_2') \Rightarrow (\neg g_1' \vee \neg g_2')$
- If both pass, all reachable states satisfy the property
  - $S \models P$

Visualizing Inductive Proofs



The inductive assertion (∼yellow) contains all initial (blue) states and no arrow leaves it (it is closed under the transition relation)

Counterexamples to Induction: The Troublemakers



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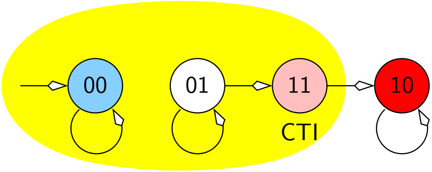
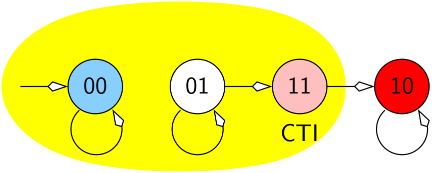


Invariant Strengthening



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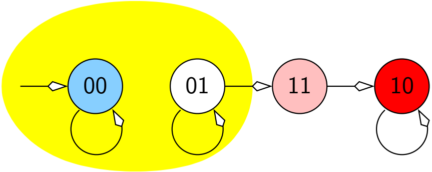

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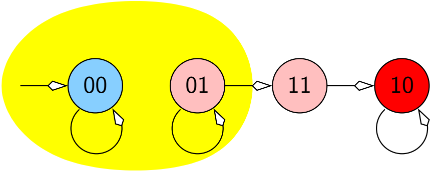

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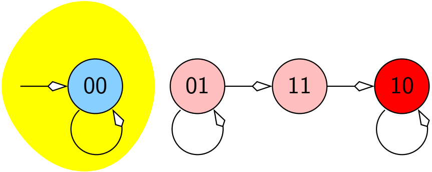
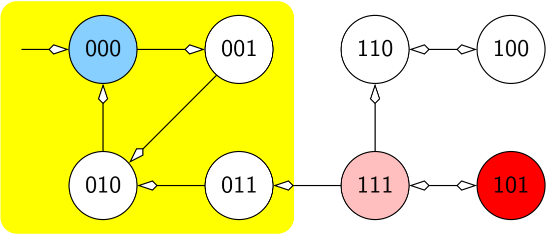


Strong and Weak Invariants



Induction is not restricted to:

- the strongest inductive invariant (forward-reachable states)
- . . . or the weakest inductive invariant (complement of the backward-reachable states)
- $\neg x_1$ is simpler than $\neg x_1 \wedge (\neg x_2 \vee \neg x_3)$ (strongest) and $(\neg x_1 \vee \neg x_3)$ (weakest)

Completeness for Finite-State Systems

- CTIs are effectively bad states
  - If a CTI is reachable so is at least one bad state
- Remove CTI from P and try again
- Eventually either:
  - An inductive strengthening of P results
  - An initial state is removed from P
- In the latter case, a counterexample is obtained

Examples of Strengthening Strategies

- Removing one CTI at a time is very inefficient!
    - Several strategies in use to avoid that
- Fixpoint-based invariant checking: if $\nu Z \,.\, p \wedge \mathrm{AX}\, Z$ converges in $n > 0$ iterations, then $\bigwedge_{0 \leq i < n} \mathrm{AX}^i\, p$ is an inductive invariant
    - In fact, the weakest inductive invariant
- k-induction: if all states on length-k paths from the initial states satisfy p, and k distinct consecutive states satisfying p are always followed by a state satisfying p, then all states reachable from the initial states satisfy p.
- fsis algorithm: try to extract an inductive clause from CTI to exclude multiple CTIs

Relative Induction

Suppose the assertion φ is a conjunction

$$\varphi = \bigwedge_{0 \leq j < n} \varphi_j$$

Suppose each φ_j is inductive relative to the previous assertions and P. That is, for every $0 \leq j < n$, $I \Rightarrow \varphi_j$ and

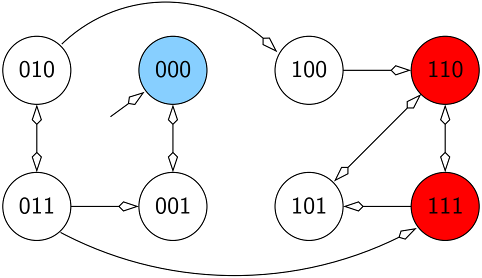
$$P \wedge \bigwedge_{0 \leq i \leq j} \varphi_i \wedge T \Rightarrow \varphi_j'$$

Finally, suppose P is inductive relative to φ; that is, $I \Rightarrow P$ and

$$P \wedge \bigwedge_{0 \leq i < n} \varphi_i \wedge T \Rightarrow P'$$

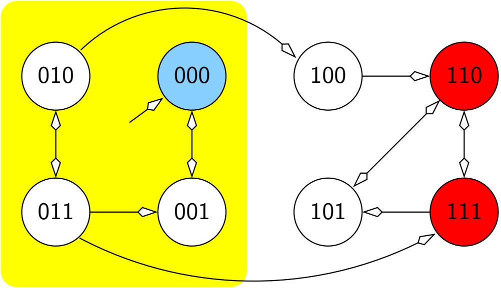
Then P is an invariant of S

Relative Induction



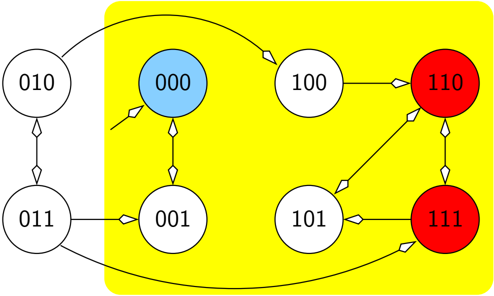
$$\varphi = \neg x_1 \wedge (x_1 \vee \neg x_2)$$

Relative Induction



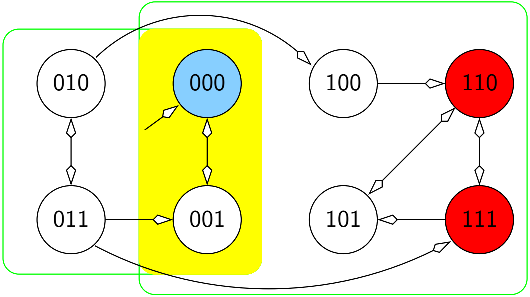
$\neg x_1$ is not inductive

Relative Induction



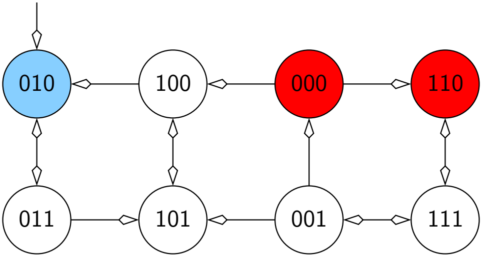
$x_1 \vee \neg x_2$ is inductive

Relative Induction



$\neg x_1$ is inductive relative to $x_1 \vee \neg x_2$

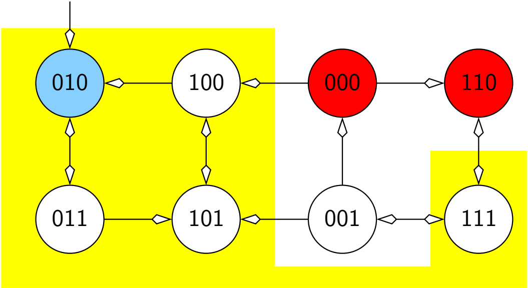
Shortcoming of Relative Induction



$$P = (x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor x_3)$$

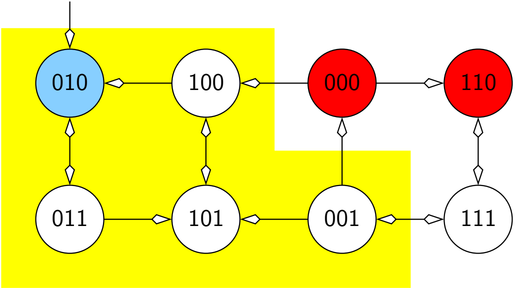
$$\varphi = (x_1 \lor x_2) \land (\neg x_1 \lor \neg x_2)$$

Shortcoming of Relative Induction



$$(x_1 \vee x_2) \wedge P \wedge T \not\Rightarrow (x_1' \vee x_2')$$

Shortcoming of Relative Induction



$$(\neg x_1 \vee \neg x_2) \wedge P \wedge T \not\Rightarrow (\neg x_1' \vee \neg x_2')$$

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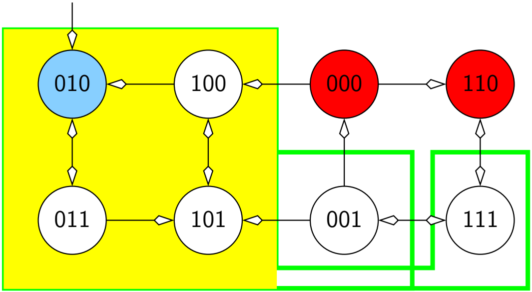
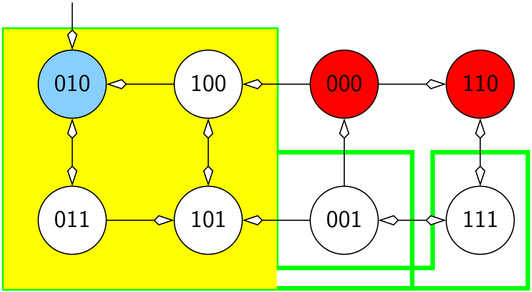


$$(x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2) \wedge P \wedge T \Rightarrow (x_1' \vee x_2') \wedge (\neg x_1' \vee \neg x_2')$$

Shortcoming of Relative Induction



$(x_1 \vee x_2)$ and $(\neg x_1 \vee \neg x_2)$are mutually inductive

Outline

What Does IC3 Stand for?

- Incremental Construction of
- Inductive Clauses for
- Indubitable Correctness

Basic Tenets

- Approximate reachability assumptions
  - F_i: contains at least all the states reachable in i steps or less
  - If $S \models P$, F_i eventually becomes inductive for some i
  - Approximation is desirable: IC3 does not attempt to get the most precise F_i's
- Stepwise relative induction
  - Learn useful facts via induction relative to reachability assumptions
- Clausal representation
  - Learn clauses from CTIs
  - A form of abstract interpretation

IC3 Invariants

- The four main invariants of IC3.

$$
\begin{aligned}
I &\Rightarrow F_0 & \\
F_i &\Rightarrow F_{i+1} & 0 \le i < k \\
F_i &\Rightarrow P & 0 \le i \le k \\
F_i \wedge T &\Rightarrow F'_{i+1} & 0 \le i < k
\end{aligned}
$$

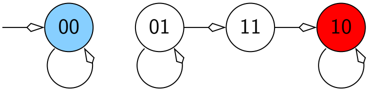
- Established if there are no counterexamples of length 0 or 1
- The implicit invariant of the outer loop: no counterexamples of length k.

Pseudo-Pseudocode

```
bool IC3 {
    if (I ⇏ P or I ∧ T ⇏ P')
        return ⊥;
    F_0 = I; F_1 = P; k = 1
    repeat {
        while (there are CTIs in F_k) {
            either find a counterexample and return ⊥
            or refine F_1, ..., F_k
        }
        k++;
        set F_k = P and propagate clauses
        if (F_i = F_{i+1} for some 0 < i < k)
            return ⊤
    }
}
```

Passing Property

No counterexamples of length 0 or 1



$$I = \neg x_1 \wedge \neg x_2$$
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$$F_i \Rightarrow F_{i+1} \qquad 0 \leq i < k$$
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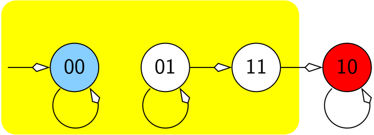
Passing Property

Does $F_1 \wedge T \Rightarrow P'$?



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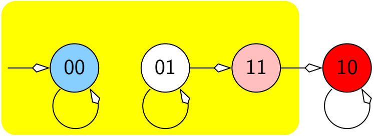
Passing Property

Found CTI $s = x_1 \wedge x_2$



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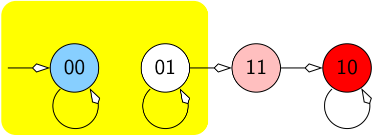
Passing Property

Is $\neg s$ inductive relative to F_1?



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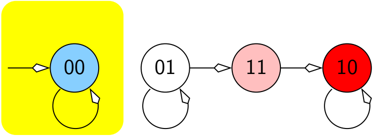
$$\begin{aligned} I &\Rightarrow F_0 & & \\ F_i &\Rightarrow F_{i+1} & & 0 \leq i < k \\ F_i &\Rightarrow P & & 0 \leq i \leq k \\ F_i \wedge T &\Rightarrow F'_{i+1} & & 0 \leq i < k \end{aligned}$$

Passing Property

No. Is $\neg s$ inductive relative to F_0?



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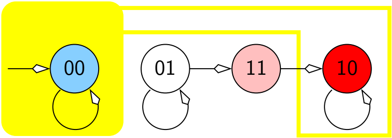
Passing Property

Yes. Generalize $\neg s$ at level 0 (in one of the two possible ways)



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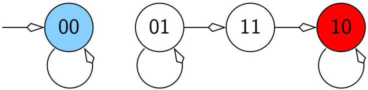
Passing Property

Update F_1



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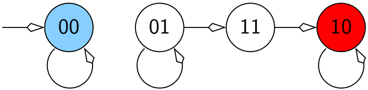
Passing Property

No more CTIs in F_1. No counterexamples of length 2. Instantiate F_2



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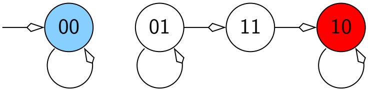
Passing Property

Propagate clauses from F_1 to F_2



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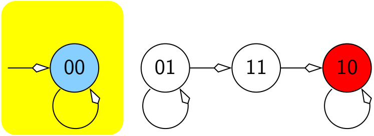
Passing Property

F_1 and F_2 are identical. Property proved



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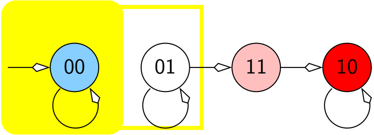
Passing Property

What happens if we generalize $\neg s$ at level 0 in the other way?



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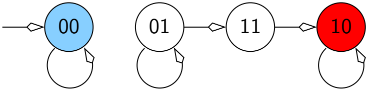
Passing Property

Update F_1



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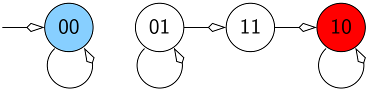
Passing Property

No more CTIs in F_1. No counterexamples of length 2. Instantiate F_2



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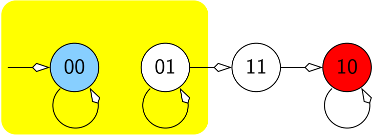
$$

Passing Property

No clauses propagate from F_1 to F_2



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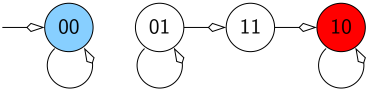
Passing Property

Remove subsumed clauses



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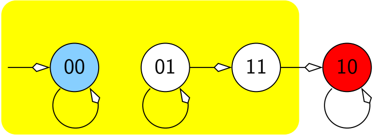
Passing Property

Does $F_2 \wedge T \Rightarrow P'$?



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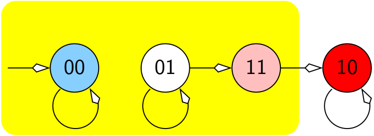
$$F_2 = P = \neg x_1 \vee x_2$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \qquad 0 \le i < k$$
$$F_i \Rightarrow P \qquad 0 \le i \le k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \le i < k$$

Passing Property

Found CTI $s = x_1 \wedge x_2$ (same as before)

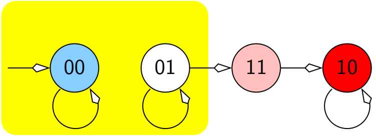


$$F_0 = I = \neg x_1 \wedge \neg x_2$$
$$F_1 = \neg x_1$$
$$F_2 = P = \neg x_1 \vee x_2$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \qquad 0 \leq i < k$$
$$F_i \Rightarrow P \qquad 0 \leq i \leq k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \leq i < k$$

Passing Property

Is $\neg s$ inductive relative to F_1?

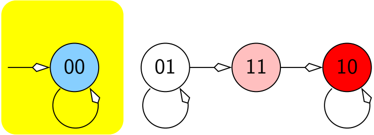


$$F_0 = I = \neg x_1 \wedge \neg x_2$$
$$F_1 = \neg x_1$$
$$F_2 = P = \neg x_1 \vee x_2$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \qquad 0 \leq i < k$$
$$F_i \Rightarrow P \qquad 0 \leq i \leq k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \leq i < k$$

Passing Property

No. We know it is inductive at level 0.

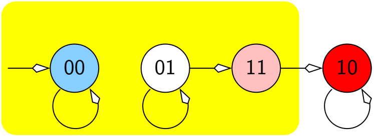


$$F_0 = I = \neg x_1 \wedge \neg x_2$$
$$F_1 = \neg x_1$$
$$F_2 = P = \neg x_1 \vee x_2$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \qquad 0 \leq i < k$$
$$F_i \Rightarrow P \qquad 0 \leq i \leq k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \leq i < k$$

Passing Property

If generalization produces $\neg x_1$ again, the CTI is not eliminated

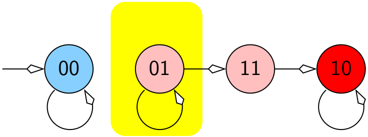


$$F_0 = I = \neg x_1 \wedge \neg x_2$$
$$F_1 = \neg x_1$$
$$F_2 = P = \neg x_1 \vee x_2$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \qquad 0 \le i < k$$
$$F_i \Rightarrow P \qquad 0 \le i \le k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \le i < k$$

Passing Property

Find predecessor t of CTI in $F_1 \setminus F_0$

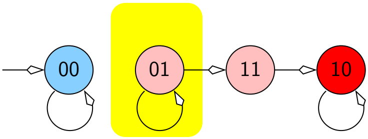


$$F_0 = I = \neg x_1 \wedge \neg x_2$$
$$F_1 = \neg x_1$$
$$F_2 = P = \neg x_1 \vee x_2$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \qquad 0 \leq i < k$$
$$F_i \Rightarrow P \qquad 0 \leq i \leq k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \leq i < k$$

Passing Property

Found $t = \neg x_1 \wedge x_2$

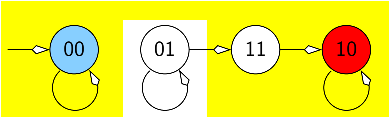


$$F_0 = I = \neg x_1 \wedge \neg x_2$$
$$F_1 = \neg x_1$$
$$F_2 = P = \neg x_1 \vee x_2$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \quad 0 \leq i < k$$
$$F_i \Rightarrow P \quad 0 \leq i \leq k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \quad 0 \leq i < k$$

Passing Property

The clause $\neg t$ is inductive at all levels

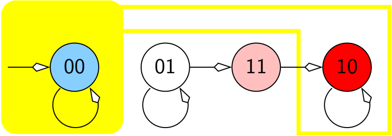


$$F_0 = I = \neg x_1 \wedge \neg x_2$$
$$F_1 = \neg x_1$$
$$F_2 = P = \neg x_1 \vee x_2$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \qquad 0 \leq i < k$$
$$F_i \Rightarrow P \qquad 0 \leq i \leq k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \leq i < k$$

Passing Property

Generalization of $\neg t$ produces $\neg x_2$

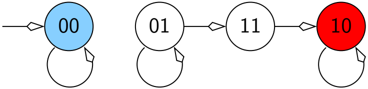


$$F_0 = I = \neg x_1 \wedge \neg x_2$$
$$F_1 = \neg x_1$$
$$F_2 = P = \neg x_1 \vee x_2$$

$$
\begin{array}{rcll}
I & \Rightarrow & F_0 & \\
F_i & \Rightarrow & F_{i+1} & 0 \leq i < k \\
F_i & \Rightarrow & P & 0 \leq i \leq k \\
F_i \wedge T & \Rightarrow & F'_{i+1} & 0 \leq i < k
\end{array}
$$

Passing Property

Update F_1 and F_2

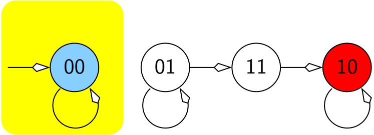


$$F_0 = I = \neg x_1 \wedge \neg x_2$$
$$F_1 = \neg x_1 \wedge \neg x_2$$
$$F_2 = (\neg x_1 \vee x_2) \wedge \neg x_2$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \qquad 0 \leq i < k$$
$$F_i \Rightarrow P \qquad 0 \leq i \leq k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \leq i < k$$

Passing Property

F_1 and F_2 are equivalent. Property (almost) proved

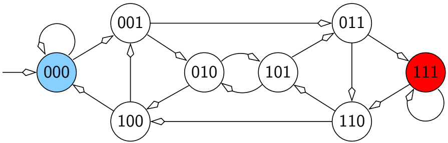


$$F_0 = I = \neg x_1 \wedge \neg x_2$$
$$F_1 = \neg x_1 \wedge \neg x_2$$
$$F_2 = (\neg x_1 \vee x_2) \wedge \neg x_2$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \qquad 0 \leq i < k$$
$$F_i \Rightarrow P \qquad 0 \leq i \leq k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \leq i < k$$

Failing Property

No counterexamples of length 0 or 1

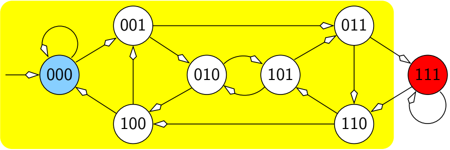


$$I = \neg x_1 \wedge \neg x_3 \wedge \neg x_3$$
$$P = \neg x_1 \vee \neg x_2 \vee \neg x_3$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \qquad 0 \leq i < k$$
$$F_i \Rightarrow P \qquad 0 \leq i \leq k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \leq i < k$$

Failing Property

Does $F_1 \wedge T \Rightarrow P'$?

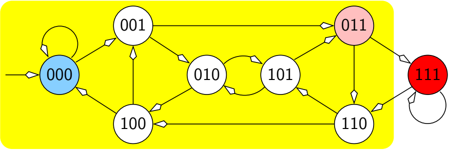


$$F_0 = I = \neg x_1 \wedge \neg x_3 \wedge \neg x_3$$
$$F_1 = P = \neg x_1 \vee \neg x_2 \vee \neg x_3$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \qquad 0 \leq i < k$$
$$F_i \Rightarrow P \qquad 0 \leq i \leq k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \leq i < k$$

Failing Property

Found CTI $s = \neg x_1 \wedge x_2 \wedge x_3$



$F_0 = I = \neg x_1 \wedge \neg x_3 \wedge \neg x_3$

$F_1 = P = \neg x_1 \vee \neg x_2 \vee \neg x_3$

$$I \Rightarrow F_0$$

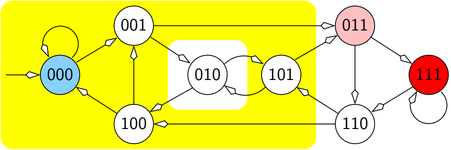
$$F_i \Rightarrow F_{i+1} \qquad 0 \leq i < k$$

$$F_i \Rightarrow P \qquad 0 \leq i \leq k$$

$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \leq i < k$$

Failing Property

The clause $\neg s$ generalizes to $\neg x_2$ at level 0

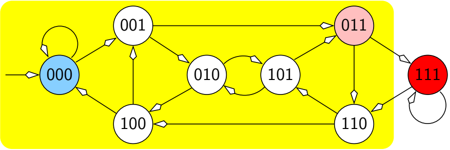


$$F_0 = I = \neg x_1 \wedge \neg x_3 \wedge \neg x_3$$
$$F_1 = (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge \neg x_2$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \qquad 0 \leq i < k$$
$$F_i \Rightarrow P \qquad 0 \leq i \leq k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \leq i < k$$

Failing Property

No CTI left: no counterexample of length 2. F_2 instantiated, but no clause propagated

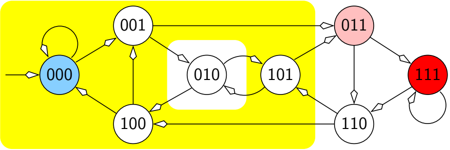


$$F_0 = I = \neg x_1 \wedge \neg x_3 \wedge \neg x_3$$
$$F_1 = \neg x_2$$
$$F_2 = P = \neg x_1 \vee \neg x_2 \vee \neg x_3$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \qquad 0 \leq i < k$$
$$F_i \Rightarrow P \qquad 0 \leq i \leq k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \leq i < k$$

Failing Property

The clause $\neg s$ generalizes again to $\neg x_2$ at level 0

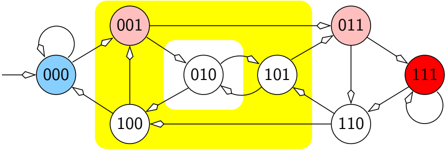


$$F_0 = I = \neg x_1 \wedge \neg x_3 \wedge \neg x_3$$
$$F_1 = \neg x_2$$
$$F_2 = P = \neg x_1 \vee \neg x_2 \vee \neg x_3$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \qquad 0 \leq i < k$$
$$F_i \Rightarrow P \qquad 0 \leq i \leq k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \leq i < k$$

Failing Property

Suppose IC3 recurs on $t = \neg x_1 \wedge \neg x_2 \wedge x_3$ in $F_1 \setminus F_0$

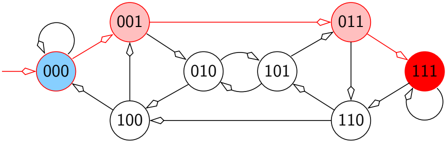


$$F_0 = I = \neg x_1 \wedge \neg x_3 \wedge \neg x_3$$
$$F_1 = \neg x_2$$
$$F_2 = P = \neg x_1 \vee \neg x_2 \vee \neg x_3$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \qquad 0 \leq i < k$$
$$F_i \Rightarrow P \qquad 0 \leq i \leq k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \leq i < k$$

Failing Property

Clause $\neg t$ is not inductive at level 0: the property fails

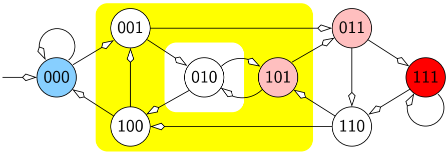


$$F_0 = I = \neg x_1 \wedge \neg x_3 \wedge \neg x_3$$
$$F_1 = \neg x_2$$
$$F_2 = P = \neg x_1 \vee \neg x_2 \vee \neg x_3$$

$$
\begin{aligned}
I &\Rightarrow F_0 & & \\
F_i &\Rightarrow F_{i+1} & & 0 \le i < k \\
F_i &\Rightarrow P & & 0 \le i \le k \\
F_i \wedge T &\Rightarrow F'_{i+1} & & 0 \le i < k
\end{aligned}
$$

Failing Property

Suppose now IC3 recurs on $t = x_1 \wedge \neg x_2 \wedge x_3$ in $F_1 \setminus F_0$

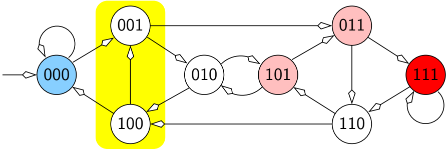


$$F_0 = I = \neg x_1 \wedge \neg x_3 \wedge \neg x_3$$
$$F_1 = \neg x_2$$
$$F_2 = P = \neg x_1 \vee \neg x_2 \vee \neg x_3$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \qquad 0 \leq i < k$$
$$F_i \Rightarrow P \qquad 0 \leq i \leq k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \leq i < k$$

Failing Property

Clause $\neg t$ is inductive at level 1

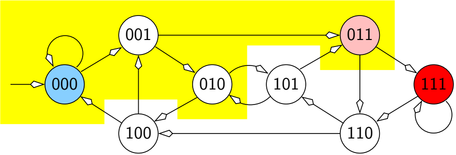


$$F_0 = I = \neg x_1 \wedge \neg x_3 \wedge \neg x_3$$
$$F_1 = \neg x_2$$
$$F_2 = P = \neg x_1 \vee \neg x_2 \vee \neg x_3$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \qquad 0 \leq i < k$$
$$F_i \Rightarrow P \qquad 0 \leq i \leq k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \leq i < k$$

Failing Property

Generalization of $\neg t$ adds $\neg x_1$ to F_1 and F_2

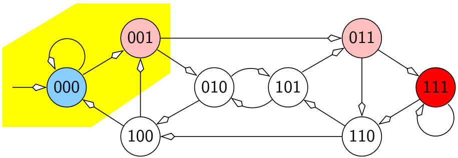


$$F_0 = I = \neg x_1 \wedge \neg x_3 \wedge \neg x_3$$
$$F_1 = \neg x_2 \wedge \neg x_1$$
$$F_2 = (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge \neg x_1$$

$$I \Rightarrow F_0$$
$$F_i \Rightarrow F_{i+1} \qquad 0 \leq i < k$$
$$F_i \Rightarrow P \qquad 0 \leq i \leq k$$
$$F_i \wedge T \Rightarrow F'_{i+1} \qquad 0 \leq i < k$$

Failing Property

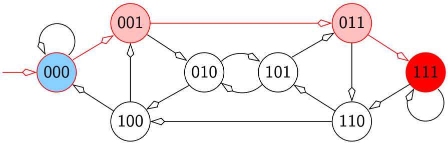
Only $t = \neg x_1 \wedge \neg x_2 \wedge x_3$ remains in $F_1 \setminus F_0$



$$
\begin{aligned}
I &\Rightarrow F_0 & \\
F_i &\Rightarrow F_{i+1} & 0 \leq i < k \\
F_i &\Rightarrow P & 0 \leq i \leq k \\
F_i \wedge T &\Rightarrow F'_{i+1} & 0 \leq i < k
\end{aligned}
$$

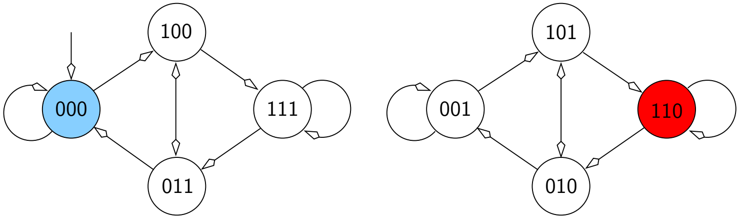
Failing Property

The same counterexample as before is found



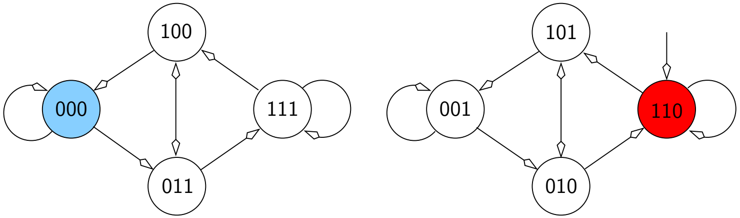
$$
\begin{aligned}
I &\Rightarrow F_0 & \\
F_i &\Rightarrow F_{i+1} & 0 \le i < k \\
F_i &\Rightarrow P & 0 \le i \le k \\
F_i \land T &\Rightarrow F'_{i+1} & 0 \le i < k
\end{aligned}
$$

Reverse IC3



Build reachability assumptions around the target

Reverse IC3



Equivalent to reversing all transitions

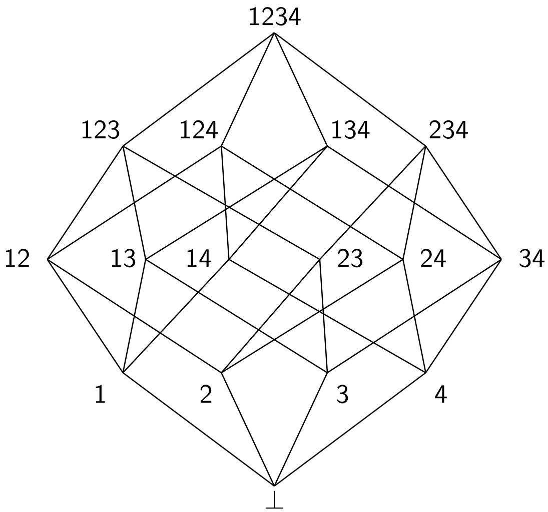
Clause Generalization

- A CTI is a cube
  - e.g., $s = x_1 \wedge \neg x_2 \wedge x_3$
- The negation of a CTI is a clause
  - e.g., $\neg s = \neg x_1 \vee x_2 \vee \neg x_3$
- Conjoining $\neg s$ to a reachability assumption F_i excludes the CTI from it
- Generalization extracts a subclause from $\neg s$ that excludes more states that are "like the CTI"
  - e.g., $\neg x_3$ may be a subclause of $\neg s$ that excludes states that, like the CTI, are not reachable in i steps
  - Every literal dropped doubles the number of states excluded by a clause
  - Generalization is time-consuming, but critical to performance

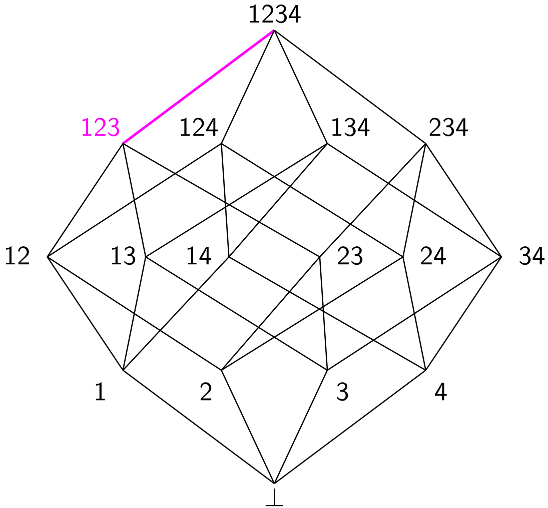
Generalization

- Crucial for efficiency
- Generalization in IC3 produces a minimal inductive clause (MIC)
- The MIC algorithm is based on DOWN and UP.
- DOWN extracts the (unique) maximal subclause
- UP finds a small, but not necessarily minimal subclause
- MIC recurs on subclauses of the result of UP

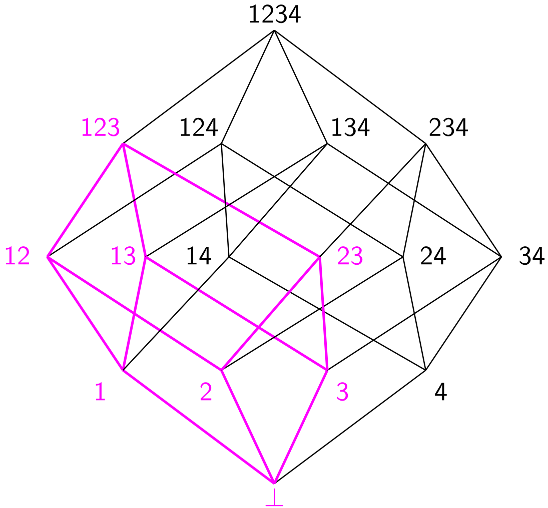
Minimal Inductive Clause



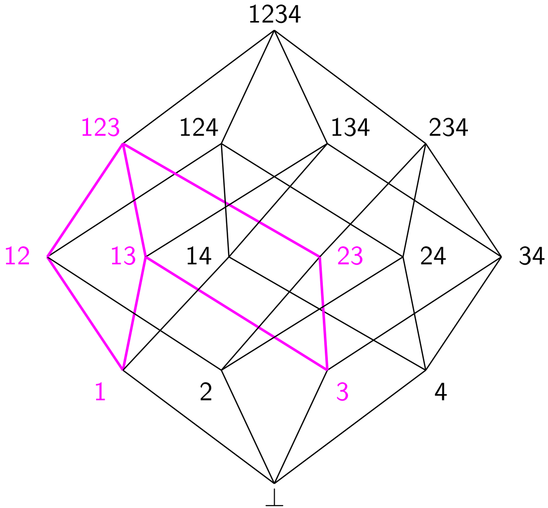
Minimal Inductive Clause



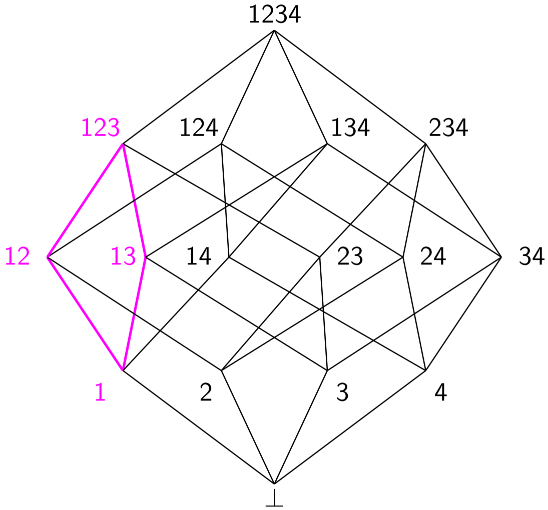
Minimal Inductive Clause



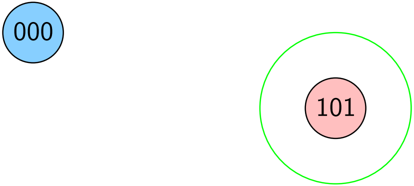
Minimal Inductive Clause



Minimal Inductive Clause

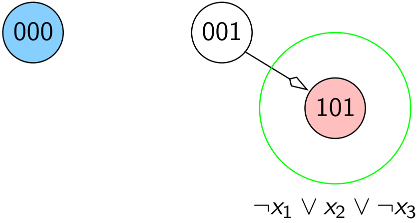


Maximal Inductive Subclause (DOWN)

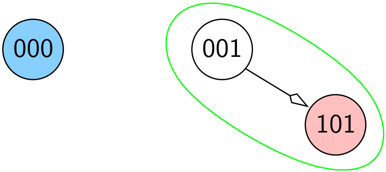


$\neg x_1 \vee x_2 \vee \neg x_3$

Maximal Inductive Subclause (DOWN)

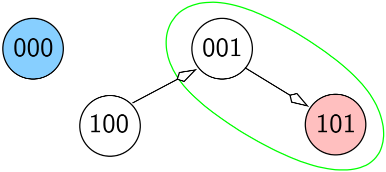


Maximal Inductive Subclause (DOWN)



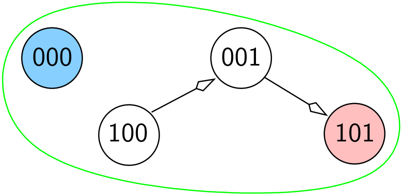
$x_2 \lor \neg x_3$

Maximal Inductive Subclause (DOWN)



$x_2 \vee \neg x_3$

Maximal Inductive Subclause (DOWN)



x_2

Use of UNSAT Cores

- $\neg s \wedge F_i \wedge T \Rightarrow \neg s'$ if and only if $\neg s \wedge F_i \wedge T \wedge s'$ is unsatisfiable
- The literals of s' are (unit) clauses in the SAT query
- If the implication holds, the SAT solver returns an unsatisfiable core
- Any literal of s' not in the core can be removed from s' because it does not contribute to the implication ...
- and from $\neg s$ because strengthening the antecedent preserves the implication

Use of UNSAT Core Example

- $\neg s \wedge F_0 \wedge T \Rightarrow \neg s'$ with

$$\begin{aligned} \neg s &= \neg x_1 \vee \neg x_2 \\ F_0 &= \neg x_1 \wedge \neg x_2 \\ T &= (\neg x_1 \wedge \neg x_2 \wedge \neg x_1' \wedge \neg x_2') \vee \cdots \end{aligned}$$

- The SAT query, after some simplification, is

$$\neg x_1 \wedge \neg x_2 \wedge \neg x_1' \wedge \neg x_2' \wedge x_1' \wedge x_2'$$

- Two UNSAT cores are

$$\begin{aligned} &\neg x_1' \wedge x_1' \\ &\neg x_2' \wedge x_2' \end{aligned}$$

from which the two generalizations we saw before follow

Clause Clean-Up

- As IC3 proceeds, clauses may be added to some F_is that subsume other clauses
- The weaker, subsumed clauses no longer contribute to the definition of F_i
- However, a weaker clause may propagate to F_{i+1} when the stronger clause does not
- Weak clauses are eliminated by subsumption only between major iterations and after propagation

More Efficiency-Related Issues

- State encoding determines what clauses are derived
- Incremental vs. monolithic
  - Reachability assumptions carry global information
  - ... but are built incrementally
- Semantic vs. syntactic approach
  - Generalization "jumps over large distances"
- Long counterexamples at low k
  - Typically more efficient than increasing k
- Consequences of no unrolling
  - Many cheap (incremental) SAT calls
- Ability to parallelize
  - Clauses are easy to exchange

IC3 and Interpolation

- An interesting analysis to be presented on Tuesday by Een, Mishchenko, and Brayton
- In the tutorial paper:
  - Both methods address the failure of consecution from an over-approximating *i*-step set.
  - Interpolation unrolls to produce an (interpolant-based) abstract post operator. When consecution fails, a greater unrolling refines the abstract post operator, yielding more refined over-approximating stepwise sets.
  - IC3 uses the CTI from the failure to direct the refinement of F_i (and $F_1, \ldots, F_{i-1}$).
  - In other words, they focus on refining **different parts of consecution**.
  - IC3 is more incremental and does not require unrolling the transition relation.

Applications

Checking all ω-regular properties

- Cycle detection reduced to several reachability queries
- Inductive proofs of unreachability refine partition of state space into SCC-closed regions

Incremental verification

- A proof from one revision of a circuit provides a starting point for the proof of the next revision
- Same for counterexample
- Some “patching” may be needed

More coming

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