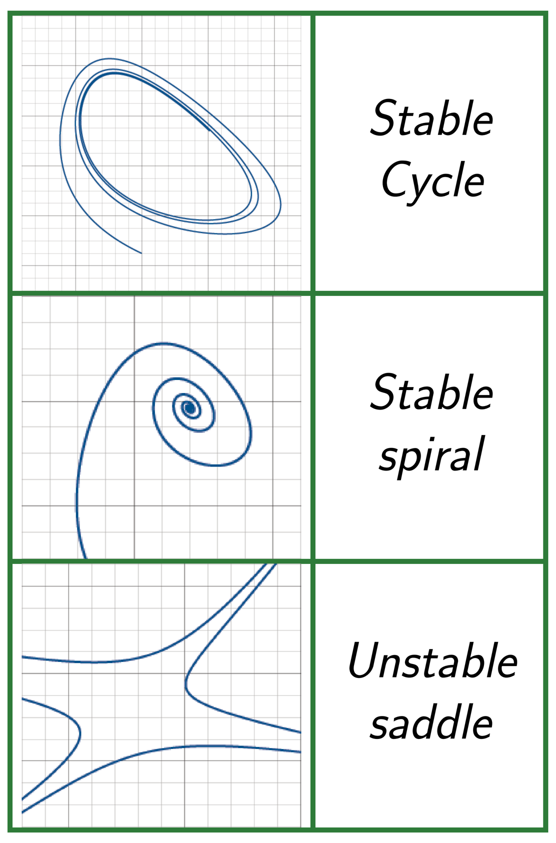
Discrete Bifurcation Analysis of Reactive Systems

Nikola Beneš, Luboš Brim, Martin Demko,
Matej Hajnal, Samuel Pastva, and David Šafránek
Masaryk University, Brno, Czech Republic



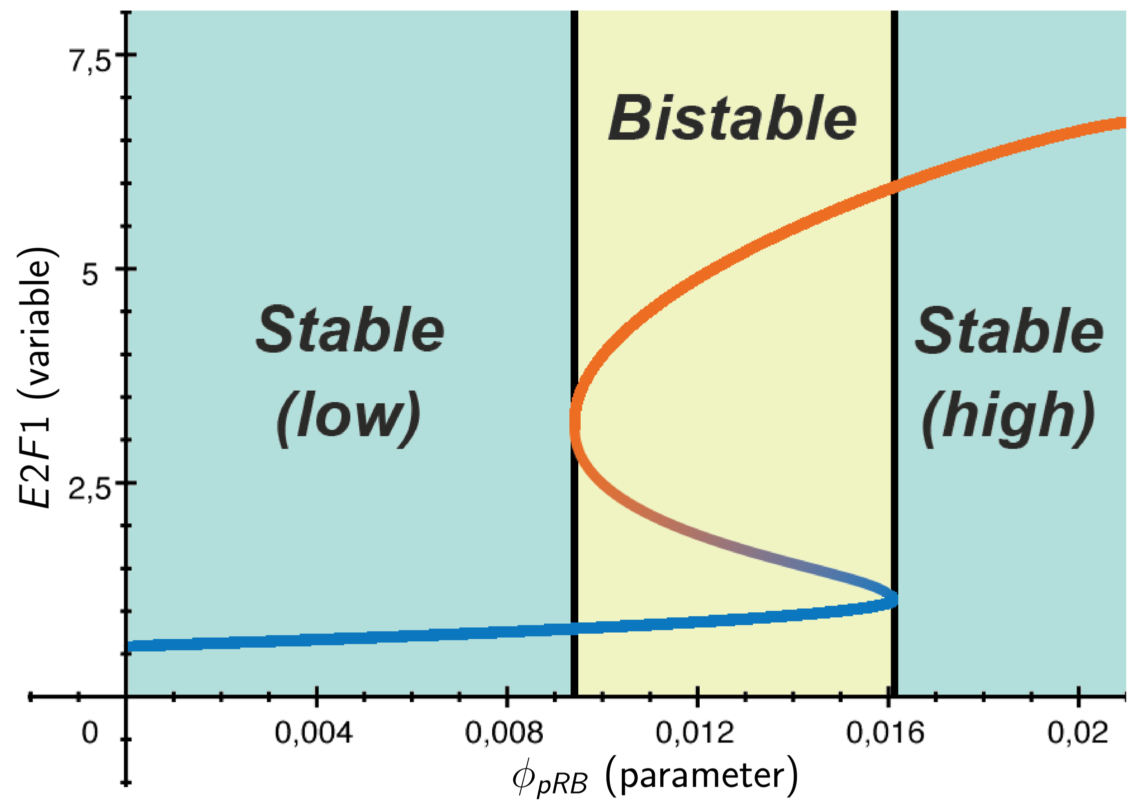
Examples of common phase portraits (patterns) as continuous trajectories.

Bifurcation Analysis

When using models to understand various phenomena, it is critical to understand how the parameters of these models affect their behaviour. Namely, when a **smooth change in parameters** results in a **qualitative change in the model behaviour**. In the study of dynamical systems, such qualitative change of behaviour is referred to as a **bifurcation** and the parameter values where this change happens are called **bifurcation points**.

A specific qualitative type of behaviour can be then associated with a corresponding phase portrait, or **pattern** of behaviour. An appearance or disappearance of these patterns manifest as bifurcation.

Current state-of-the-art techniques for bifurcation analysis are often very **restrictive** (requiring specific types of models), **hard to automate and parallelise**, impossible or **difficult to scale** in the number of parameters, and applicable only to **models based on differential or difference equations**.



Bifurcation diagram of our example model, showing the location of equilibria with respect to the model's parameter, together with the bifurcation points (black lines) and the patterns of behaviour (text).

Temporal Logic as Pattern Specification Language

The notion of behavioural patterns can be captured using **temporal logic formulae**. To this end, we propose a **hybrid computation tree logic with past** ($HUCTL_P$) [2], based on CTL with the following extensions:

- **Hybrid operators bind**, **at**, **exists**, and **forall** in order to enable qualitative reasoning.
- **A sense of direction** $_{x+}\mathbf{F}\varphi$ in path operators to allow reasoning about monotonicity and various flow properties.
- **Backward state operators** $\hat{\mathbf{E}}$ and $\hat{\mathbf{A}}$ (aside from classic forward **E** and **A**) to reason both about future and past evolution of the system.

The problem of finding the bifurcation points of a system can be then studied via **parameter synthesis** for various types of behavioural patterns.

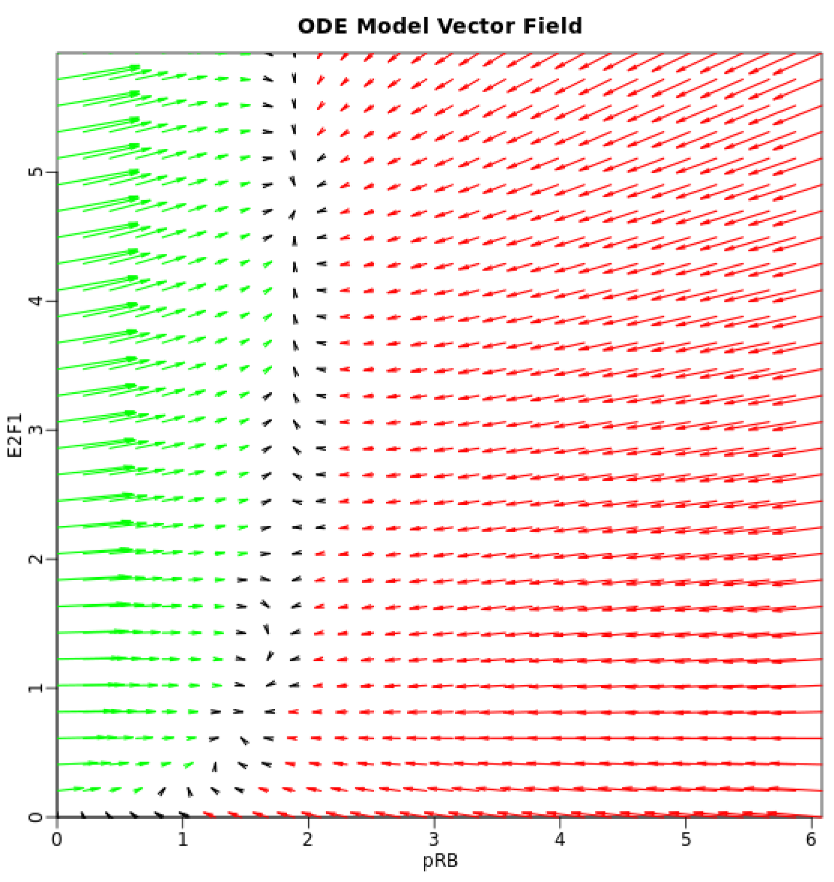
steady State x is an equilibrium (either stable or unstable).
[**bind** x: **EX** x]

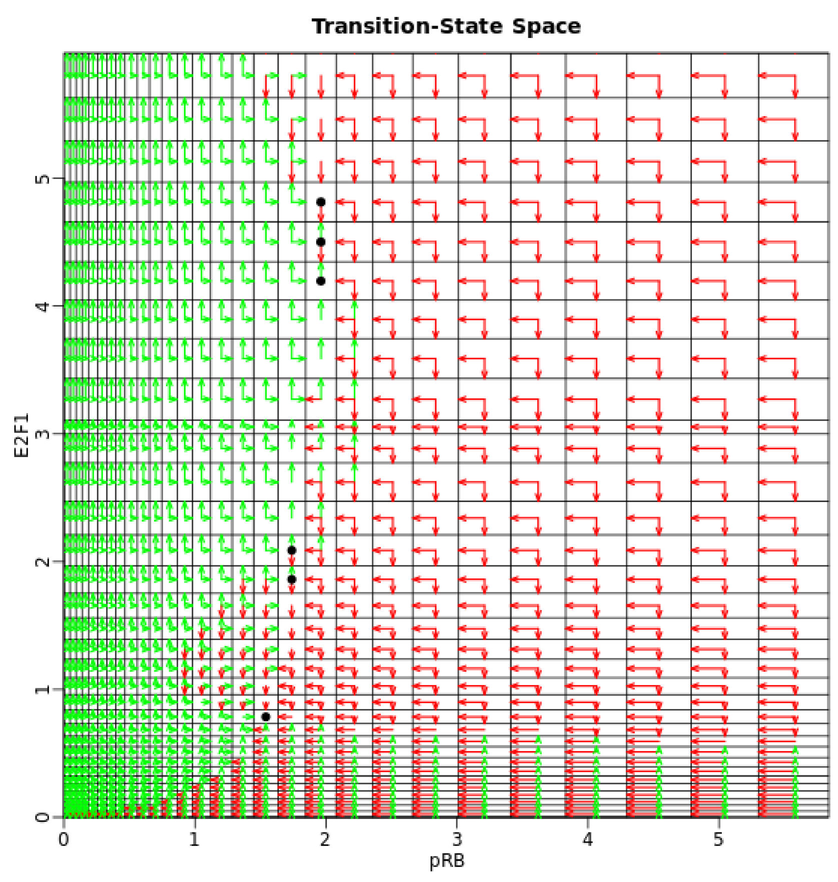
source State x is an unstable source.
[**bind** x: $\hat{\mathbf{A}}$**X** x]

cycle State x lies on a cycle (not necessarily stable).
[**bind** x: **EX EF** x]

stable State x lies in a stable component.
[**bind** x: **AG EF** x]

bistable There are at least two disconnected stable components.
[**exists** s **in stable: exists** t **in stable: at** t: **not EF** s]



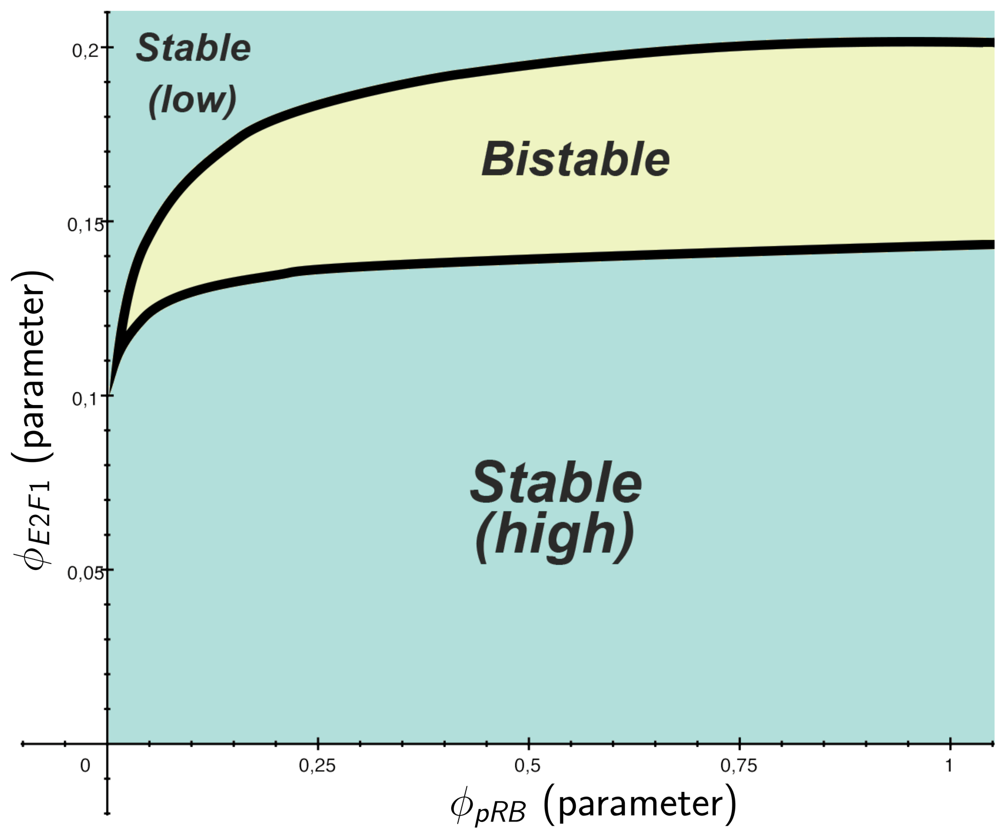


From Differential Equations to Reactive Systems

As an input to the parameter synthesis method, we consider a **parametrised direction transition system** (pDTS). pDTS is a doubly labelled transition system, where each transition has an assigned direction and a set of parameter values under which it is enabled. We represent these parameter sets using **SMT formulae**, thus creating an efficient semi-symbolic structure as described in [3], implemented in our parameter synthesis tool **PITHYA** [1].

A pDTS can be then used to describe the dynamics of various types of reactive systems. As an example (and to compare with the continuous bifurcation analysis), we focus on an abstraction of systems based on autonomous differential equations [4]. For piece-wise multi-affine systems, this approach provides a safe over-approximation of the model.

An example of the continuous phase space together with the discrete abstract transition system is shown on the left.



Discrete bifurcation diagram plotting the dependence of stable/bistable patterns in the two-parametric case.

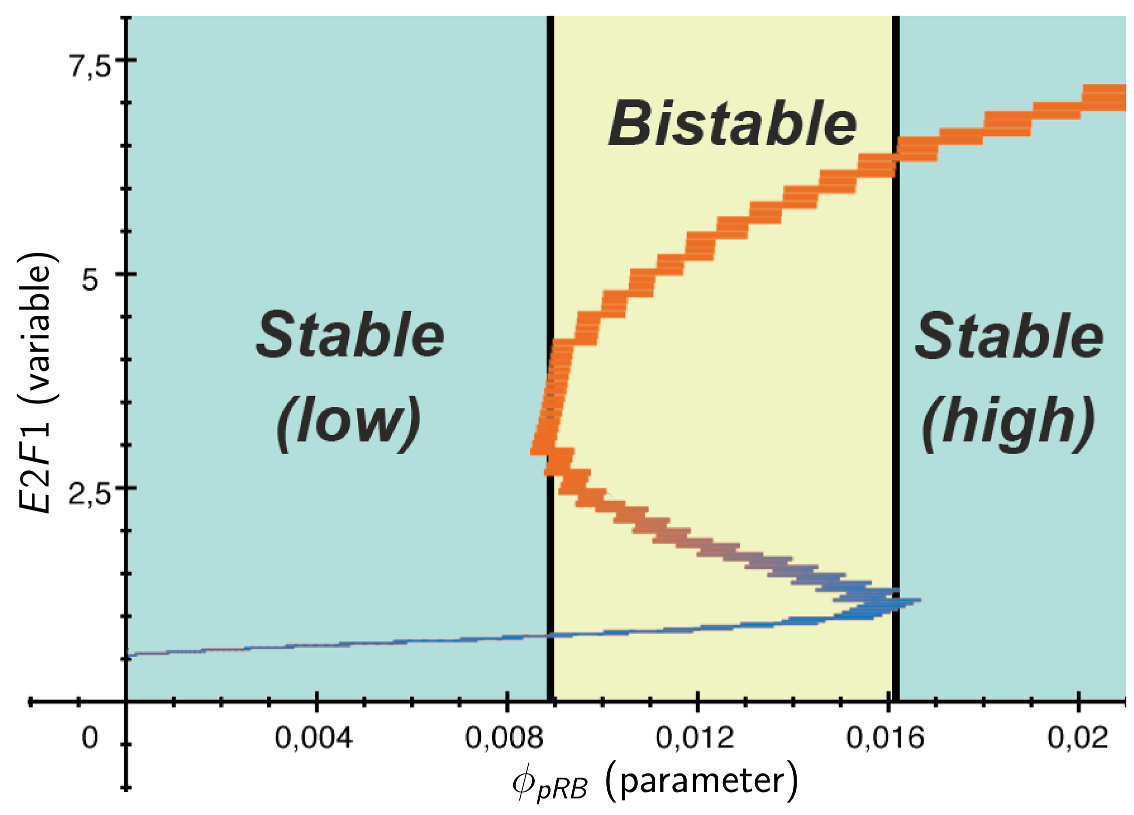
Case Study in Discrete Bifurcation Analysis

To show the applicability of our method, we consider an abstraction of a G_1/S **bistable mammalian switch** model [5]. We take this model either with a **single parameter** (ϕ_{E2F1} fixed to 0.1) or with **two unknown parameters** (ϕ_{pRB} and ϕ_{E2F1}).

On the right, we show that the **steady states** and the **stable and bistable patterns** in the single-parametric model follow the locations in the **original continuous model**.

Second, **on the left**, we perform a more detailed analysis of the system, visualizing the **dependence of stable and bistable patterns on both parameters** of the model.

In these diagrams, the bifurcation points are shown as black lines. The type of the stable state (low or high) is based on its location in the model variable $E2F1$.



Discrete bifurcation diagram, plotting the location of steady states depending on the parameter value.

References

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